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A NOTE ON LOCAL FUNCTION IN GENERALIZED IDEAL TOPOLOGICAL SPACES

J. K. MAITRA, RAJESH TIWARI & V. TIWARI

Department of Mathematics and Computer Science, Rani Durgawati University, Jabalpur, Madhya Pradesh, India

ABSTRACT

In this article, we have developed the theory of local function of a set in the context of generalized ideal topological space. Further we have constructed some basic examples.

KEYWORDS: Generalized Topological Spaces, Generalized Ideal Topological Spaces, Local Function

1. INTRODUCTION

The concept of ideal in Topological spaces was introduced by Kuratowski [4] in 1930. The notion of generalized topology was introduced by Csaszar [1] in 2002. Jancovic and Hamlett [3] have studied the concept of local function in ideal topological spaces and obtained its significant properties. The concept of Ideal in generalized topological spaces is defined by Maitra and Tripathi [5] in 2014. They have obtained important properties of local function in generalized ideal topological spaces.

2. PRELIMINARIES

First we recall the definition of generalized topological space, g-open sets and g-closed sets.

Definition 2.1: [5] Let X be a non empty set and let τ_g be a family of subsets of X. Then τ_g is said to be a **generalized topology** on X if following two conditions are satisfied viz,:

$\phi,\ X \in \tau_g$;

If $G_\lambda \in \tau_g$ for $\lambda \in \wedge$ then $\bigcup_{\lambda\in\wedge} G_\lambda \in \tau_g$.

The pair (X, τ_g) is called a **generalized topological space**. The members of the family τ_g are called **g-open sets** and their complements are called **g-closed sets**.

From the above Definition 2.1 we observe that every topological space is a generalized topological space but the converse is not true. We have following Example.

Example 2.1: Let $X = \{a, b, c\}$ and let $\tau_g = \{\emptyset, X, \{a, b\}, \{b, c\}\}$. Then τ_g is a generalized topology but not a topology on X.

Definition 2.2: Let (X, τ_g) be a generalized topological space and $A \subseteq X$. Then **τ_g-interior** of A is denoted by $Int_{\tau_g}(A)$ and is defined to be the union of all g-open sets in X contained in A. The **τ_g-closure** of A is denoted by $Cl_{\tau_g}(A)$ and is defined to be the Intersection of all g-closed sets in X containing A.

Remark: Since arbitrary union of g-open sets is a g-open set and arbitrary intersection of g-closed sets is a g-closed sets, it follows that $Int_{\tau_g}(A)$ is a g-open set and $Cl_{\tau_g}(A)$ is a g-closed set. Thus $Int_{\tau_g}(A)$ is the largest g-open set contained in A and $Cl_{\tau_g}(A)$ is the smallest g-closed set containing A.

Proposition 2.1: Let (X, τ_g) be a generalized topological space and $A \subseteq X$. Then

- A is a g-open set iff $Int_{\tau_g}(A) = A$.
- A is a g-closed set iff $Cl_{\tau_g}(A) = A$.

Proof

- Suppose A is g-open set in X. Since $Int_{\tau_g}(A)$ is the union of all g-open sets in X contained in A and $A \subseteq A$ it follow that $A \subseteq Int_{\tau_g}(A)$. As we know that $Int_{\tau_g}(A) \subseteq A$, we have, $Int_{\tau_g}(A) = A$. Conversely, suppose $Int_{\tau_g}(A) = A$. Then by definition of τ_g-interior of A, we note that $Int_{\tau_g}(A)$ is a g-open set. Thus A is a g-open set in X.
- Suppose A is g-closed set in X. Since $Cl_{\tau_g}(A)$ is the intersection of all g-closed sets in X containing A and $A \subseteq A$ it follow that $Cl_{\tau_g}(A) \subseteq A$. As we know that $A \subseteq Cl_{\tau_g}(A)$, we have, $Cl_{\tau_g}(A) = A$. Conversely, suppose $A = Cl_{\tau_g}(A)$. Then by definition of τ_g-closure of A, we note that $Cl_{\tau_g}(A)$ is a g-closed set. Thus A is a g-closed set in X.

Theorem 2.1: Let (X, τ_g) be a generalized topological space and $A \subseteq X$. Then $x \in Cl_{\tau_g}(A)$ iff each g-open set $U \in \tau_g$ containing x intersects A.

Proof: Let (X, τ_g) be a generalized topological space and $A \subseteq X$. Suppose $x \in Cl_{\tau_g}(A)$ and U is a g-open set in X such that $x \in U$. Then $X - U$ is a g-closed set and $x \notin X - U$. As $\in Cl_{\tau_g}(A)$, we must have $A \nsubseteq X - U$. This means there exists an element $a \in A$ and $a \notin X - U$, i.e., $a \in U$. Thus U intersects A.

Conversely suppose that each g-open set $U \in \tau_g$ containing x intersects A. Let F be a g-closed set containing A. Then $X - F$ is a g-open set disjoint from A. From hypothesis $x \notin X - F$, i.e., $x \in F$. Thus x belongs to each g-closed set containing A and so $x \in Cl_{\tau_g}(A)$.

Theorem 2.2: Let (X, τ_g) be a generalized topological space and $A \subseteq X$. Then $x \in Int_{\tau_g}(A)$ iff there exists a g-open set $U \in \tau_g$ such that $x \in U \subseteq A$.

Proof: Let (X, τ_g) be a generalized topological space and $A \subseteq X$. Suppose $x \in Int_{\tau_g}(A)$. Since $Int_{\tau_g}(A)$ is a union of all g-open sets contained in A, there exists a g-open set $U \in \tau_g$ such that $x \in U \subseteq A$.

Conversely suppose that $x \in X$ and there exists a g-open set $U \in \tau_g$ such that $x \in U \subseteq A$. Since $Int_{\tau_g}(A)$ is the largest g-open set contained in A, we have $U \subseteq Int_{\tau_g}(A)$. Thus $x \in Int_{\tau_g}(A)$.

Definition 2.3: [5] Let (X, τ_g, I) be a generalized ideal topological space and $A \subseteq X$. The set $A^\star(I) = \{x \in X: A \cap U \notin I$ for each g-open set U containing $x\}$ is called **local function** of A with respect to Ideal I and generalized topology τ_g

on X.

Example 2.2: Let $X = \{a, b, c\}$, $\tau_g = \{\emptyset, X, \{a, b\}, \{b, c\}\}$ be generalized topology on X and $I = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$ be ideal on X. Then (X, τ_g, I) is a generalized ideal topological space. We see that local function of each subsets of X are as follows:

- $\emptyset^\star = \emptyset$
- $\{a\}^\star = \emptyset$
- $\{b\}^\star = \emptyset$
- $\{c\}^\star = \{c\}$
- $\{a, b\}^\star = \emptyset$
- $\{a, c\}^\star = \{c\}$
- $\{b, c\}^\star = \{c\}$
- $X^\star = \{c\}$

Proposition 2.2: [5] Let (X, τ_g, I) be a generalized ideal topological space and A, B be subsets of X.

- If $A \subseteq B$ then $A^\star \subseteq B^\star$
- $A^\star \cup B^\star \subseteq (A \cup B)^\star$
- $(A^\star)^\star \subseteq A^\star$
- $A^\star$ is a g-closed set in (X, τ_g)

We have verified above properties by following examples:

Example 2.3: Let $X = \{a, b, c\}$ be the generalized ideal topological space with respect to generalized topology $\tau_g = \{\emptyset, X, \{a, b\}, \{b, c\}\}$ and ideal $I = \{\emptyset, \{b\}\}$ on X. Let us consider $A = \{c\}$ and $B = \{a, c\}$. Then $A^\star = \{c\}$ and $B^\star = X$. We see that property (i) holds in Proposition 2.2.

Example 2.4: Let $X = \{a, b, c, d\}$ be the generalized ideal topological space with respect to generalized topology $\tau_g = \{\emptyset, X, \{a, b, d\}, \{b, c\}\}$ and ideal $I = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$ on X. Let us consider $A = \{c\}$ and $B = \{d\}$. Then we have $A^\star = \{c\}$, $B^\star = \{a, d\}$ and $(A \cup B)^\star = X$. We see that property (ii) holds in Proposition 2.2.

Proposition 2.3: [5] Let (X, τ_g) be a generalized topological space and I_1, I_2 be two ideals on. X. If $I_1 \subseteq I_2$ then

$A^\star(I_2) \subseteq A^\star(I_1)$.

In the following example we have verified the above result.

Example 2.5: Let $X = \{a, b, c\}$ be the generalized topological space with respect to generalized topology $\tau_g = \{\emptyset, X, \{a, b\}, \{b, c\}\}$. Let us consider ideals $I_1 = \{\emptyset, \{a\}\}$ and $I_2 = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$ on X. For set $A = \{\text{b}\}$, we have $A^\star(I_1) = X$ and $A^\star(I_2) = \emptyset$. Thus we see that Proposition 2.2 holds.

3. SIGNIFICANT RESULTS OF LOCAL FUNCTION IN GENERALIZED IDEAL TOPOLOGICAL SPACE

In this section we have observed significant properties of local function in generalized ideal topological spaces. We have also constructed some important examples.

Theorem 3.1: Let (X, τ_g, I) be a generalized ideal topological space. If $I = \wp(X)$, then $A^\star = \emptyset$, for each subset A of X.

Proof: Suppose $x \in X$ and U is a g-open set in X containing x. Then for each subset A of X, $U \cap A \in I$. This implies $x \notin A^\star$. Hence $A^\star = \emptyset$.

In above Example 2.1, we see that $X^\star \subseteq X$. In the following result we get a necessary and sufficient condition for $X^\star = X$.

Theorem 3.2: Let (X, τ_g, I) be a generalized ideal topological space. Then $I \cap \tau_g = \phi$ iff $X^* = X$.

Proof: Let (X, τ_g, I) be a generalized ideal topological space and $I \cap \tau_g = \emptyset$. Suppose $x \in X$ and U is a g-open set in X such that $x \in U$. Then $U \cap X = U$. Since $U \neq \emptyset$, it follow that $U \notin I$. Thus $x \in X^\star$. Therefore $X^\star = X$.

Conversely suppose $X^\star = X$. Let $U \in \tau$ and $U \neq \emptyset$. Then there exits $x \in X = X^\star$, such that $x \in U$. This means U is g-neighborhood of x. Now by definition of $X^\star$, $U \cap X = U \notin I$. Thus each non-empty g-open set in X can not lie in I. Hence $I \cap \tau = \emptyset$.

Theorem 3.3: Let X be a generalized topological space and I, J be two ideals on set X. Then $A^\star(I) \cup A^\star(J) \subseteq A^\star(I \cap J)$.

Proof: Since $I \cap J \subseteq I$ from Proposition 2.2, $A^\star(I) \subseteq A^\star(I \cap J)$. Similarly $A^\star(J) \subseteq A^\star(I \cap J)$. Hence we have $A^\star(I) \cup A^\star(J) \subseteq A^\star(I \cap J)$.

Remark: In the above Theorem 3.3 $A^\star(I) \cup A^\star(J) \neq A^\star(I \cap J)$ in general, we have following example.

Example 3.1: Let $X = \{a, b, c, d\}$ be the generalized topological space with respect to generalized topology $\tau_g = \{\emptyset, X, \{a, b, c\}, \{a, b, d\}, \{c, d\}\}$. Let us take ideals $I = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$ and $J = \{\emptyset, \{c\}, \{d\}, \{c, d\}\}$ on X. Consider set $A = \{a, c\}$. We have $A^\star(I) = \{c\}$, $A^\star(J) = \{a, b\}$ and $A^\star(I \cap J) = X$. Thus we see that $A^\star(I) \cup A^\star(J) \neq A^\star(I \cap J)$.

Proposition 3.1: Let (X, τ_g, I) be a generalized ideal topological space and $A \subseteq X$. Then $A^\star \subseteq Cl_{\tau_g}(A)$.

Proof: Let $A \subseteq X$ and $x \in A^\star$. Suppose F is a g-closed set in (X, τ_g, I) and $A \subseteq F$. Then $(X - F) \subseteq (X - A)$. If possible assume that $x \in X - F$, then $(X - F) \cap A \notin I \Rightarrow (X - F) \cap A \neq \emptyset \Rightarrow (X - A) \cap A \neq \emptyset$ which is a contradiction.

Hence $x \notin (X - F)$ ie. $x \in F$. Thus x belongs to each closed set containing A, ie. $x \in Cl_{\tau_g}(A)$. Hence $A^\star \subseteq Cl_{\tau_g}(A)$.

Corollary 3.1: Let (X, τ_g, I) be a generalized ideal topological space. If $I = \emptyset$ then $A^\star = Cl_{\tau_g}(A)$.

Proof: Let (X, τ_g, I) be a generalized ideal topological space and let $A \subseteq X$. If $x \in Cl_{\tau_g}(A)$ then each g-nbd U of x intersects A, i.e. $U \cap A \neq \emptyset \Rightarrow U \cap A \notin I$. Hence $x \in A^\star$. Thus we find that $Cl_{\tau_g}(A) \subset A^\star$. Since from above result $A^\star \subseteq Cl_{\tau_g}(A)$, we have $A^\star = Cl_{\tau_g}(A)$.

Corollary 3.2: Let (X, τ_g, I) be a generalized ideal topological space and let A be g-closed subset of X. Then $A^\star \subseteq A$.

Proof: From Proposition 3.1 $A^\star \subseteq Cl_{\tau_g}(A)$. Since A is a g-closed set, we have $Cl_{\tau_g}(A) = A$. This implies, $A^\star \subseteq A$.

Example 3.2: Let $X = \{a, b, c, d\}$ be the generalized ideal topological space with respect to generalized topology $\tau_g = \{\emptyset, X, \{a, b, d\}, \{b, c\}\}$ and ideal $I = \{\emptyset, \{c\}\}$ on X. Consider set $A = \{c\}$. Then we have $A^\star = \emptyset$. Thus we see that $A^\star \subseteq A$.

Corollary 3.3: Let (X, τ_g, I) be a generalized ideal topological space and let A be g-closed subset of X. If $I = \{\emptyset\}$ then $A^\star = A$.

Proof: Let (X, τ_g, I) be a generalized ideal topological space and $A \subseteq X$. Since $I = \{\emptyset\}$ by Corollary 3.1, $A^\star = Cl_{\tau_g}(A)$. As A is a g-closed set, we have $A = Cl_{\tau_g}(A)$. This means $A^\star = A$.

In the following example we observe that neither a set is contained in its local function nor a local function of set is contained in the set.

Example 3.3: Let $X = \{a, b, c\}$ be the ideal topological space with respect to topology $\tau_g = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$ and ideal $I = \{\emptyset, \{a\}\}$ on X. Suppose $A = \{a, b\}$. Then we have $A^\star = \{b, c\}$. Thus we see that neither A is contained in $A^\star$ nor $A^\star$ is contained in A.

Theorem 3.4: Let (X, τ_g, I) be a generalized ideal topological space and $A \subseteq X$. Then $A^\star = Cl_{\tau_g}(A^\star) \subseteq Cl_{\tau_g}(A)$.

Proof: Since $A^\star$ is a g-closed set in (X, τ_g, I) (Proposition 2.1), we have , $A^\star = Cl_{\tau_g}(A^\star)$. Further from Proposition 3.1, $A^\star \subseteq Cl_{\tau_g}(A)$. This implies $Cl_{\tau_g}(A^\star) \subseteq Cl_{\tau_g}\left(Cl_{\tau_g}(A)\right) = Cl_{\tau_g}(A)$. Thus $Cl_g(A^\star) \subseteq Cl_{\tau_g}(A)$.

Example 3.4: Let $X = \{a, b, c, d\}$ be the generalized ideal topological space with respect to generalized topology $\tau_g = \{\emptyset, X, \{a, b\}, \{b, c\}, \{a, b, c\}\}$ and ideal $I = \{\emptyset, \{b\}, \{c\}, \{b, c\}\}$ on X. Suppose $A = \{a, b\}$. Then we have $A^\star = \{a, d\}$, $Cl_{\tau_g}(A^\star) = \{a, d\}$ and $Cl_{\tau_g}(A) = X$. Thus we see that $A^\star = Cl_{\tau_g}(A^\star) \subseteq Cl_{\tau_g}(A)$.

In generalized ideal topological space for $A \subseteq X$, we associate the subset denoted as $Cl_{\tau_g}{}^\star(A)$ of X, and is defined as $Cl_{\tau_g}{}^\star(A) = A \cup A^\star$, where $A^\star$ is local function of set A.

Proposition 3.2: Let (X, τ_g, I) be a generalized ideal topological space and $A \subseteq X$.Then $Cl_{\tau_g}{}^\star(A) \subseteq Cl_{\tau_g}(A)$.

Proof: Since ${Cl_{\tau_g}}^\star(A) = A \cup A^\star$ and $A^\star \subseteq Cl_{\tau_g}(A)$, it follows that $cl^\star(A) \subseteq A \cup Cl_{\tau_g}(A) = Cl_{\tau_g}(A)$. Thus ${Cl_{\tau_g}}^\star(A) \subseteq Cl_{\tau_g}(A)$.

Proposition 3.3: Let (X, τ_g, I) be a generalized ideal topological space and $A \subseteq X$. If $I = \{\emptyset\}$ then ${Cl_{\tau_g}}^\star\,(A) = Cl_{\tau_g}(A)$.

Proof: Since $Cl^\star_{\tau_g}(A) = A \cup A^\star$, from Corollary 3.1, we have $Cl^\star_{\tau_g}(A) = A \cup Cl_{\tau_g}(A) = Cl_{\tau_g}(A)$. Thus ${Cl_{\tau_g}}^\star\,(A) = Cl_{\tau_g}(A)$.

Example 3.5: Let $X = \{a, b, c, d\}$ be the ideal topological space with respect to topology $\tau_g = \{\emptyset, X, \{a, b\}, \{c, d\}\}$ and ideal $I = \{\emptyset\}$ on X. Suppose $A = \{a, b\}$. Then we have $A^\star = \{a, b\}$, $Cl_{\tau_g}(A^\star) = \{a, b\}$ and $Cl_{\tau_g}(A) = \{a, b\}$. Thus we see that $Cl_{\tau_g}(A) = {Cl_{\tau_g}}^\star(A)$.

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